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I-GENERALIZED METRIC SPACES AND SOME FIXED POINT THEOREMS

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Abstract
The aim of this paper is to establish the structure of I-generalized metric spaces as a concept of metric spaces, which is a kind of generalization of traditional generalized metric space structure. Some fixed point results for various contractive type mappings in the context of I-generalized metric spaces are presented. We also provide some definitions to illustrate the results presented herein.

Mathematics Subject Classification: 47H10, 54H25

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1. Introduction

The concept of metric spaces has been generalized in many directions. The notion of a metric space induces topological properties like open and closed sets which leads to the study of more abstract topological spaces.

In 2000, Branciari [4] introduced generalized metric spaces replacing triangular inequality by rectangular inequality and subsequently several fixed point results have been developed in this metric space.

A generalization of contraction mapping is in Banach contraction [4], uniformly locally contraction [5], Kannan contraction [8] mappings in generalized metric space.

Here we shall generalize these concepts and some more fixed point results [2] in I-generalized metric space.

2. Preliminaries

First we recall some notation and definitions that will be utilized in our subsequent discussion.

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Definition (2.1) [I-uniqueness or I-equality] Let X be a non-empty set and $f : X \to X$ be an idempotent map. Two elements x and y in X are said to be I-unique with respect to f if $f(x) = f(y)$; otherwise x and y are said to be I-distinct points in X.

Definition (2.2)[I-generalized metric space] Let X be a non-empty set, $f: X \to X$ be an idempotent map, *i.e.* , $f^2 = f$. *A* map $d : X^2 \to [0, \infty)$ is said to be an I-generalized metric on X iff
I_1: $\forall x, y \in X,\ d(x, f(y)) = 0\ iff\ f(x) = f(y)$ and $d(f(x), y) = 0\ iff\ f(x) = f(y)$.
I2: $d(x, f(y)) = d(y, f(x))$ and $(f(x), y) = d(f(y), x), \forall x, y \in X$.
I_3: for all $x, y \in X$ and for all I-distinct points $u, v \in X$ each of them I-distinct from x and y, $(x, y) \leq d(f(x), u) + d(f(u), v) + d(v, f(y))$.
The order triple (X, d, f) is called an I-metric space. Elements of X are said to be points in X .

Example (2.3): (i) Every I-metric space is clearly a I-g.m.s.
(ii) Every generalized metric space (X, d) is clearly a I-g.m.s. with respect to the identity map on X.

Definition (2.4) [I-open sphere] Let (X, d, f) be an I-metric space and $x \in X$ and r be a positive real number. Then the set $S_f(x, r) = \{y \in X \mid d(x, f(y)) < r\}$ is called the I-open sphere or I-open ball, with centre x and radius r in X.

Definition (2.5) [Convergence of a sequence] A sequence $\{x_n\}$ in an I-metric space (X, d, f) is said to I-converge to a point x of X , if for any $\varepsilon > 0, \exists\, m \in \mathbb{N}$ such that $x_n \in S_f(x, \varepsilon), \forall n \geq m$. In this case x is called I-limit of the sequence $\{x_n\}$.
A sequence which is not I-convergent in an I-metric space (X, d, f), is called a non-I-convergent or an I-divergent sequence.

Definition (2.6) [Cauchy sequence] A sequence $\{x_n\}$ in an I-metric space (X, d, f) is said to be an I-cauchy sequence in X if for any $\varepsilon > 0, \exists\, n_o \in \mathbb{N}$ such that $d(f(x_m), x_n) < \varepsilon, \forall m, n \geq n_o$, *i.e.* , $d(f(x_{n+p}), x_n) < \varepsilon$, $\forall\, n \geq n_o, \forall p \geq 1$.

Definition (2.7) [Complete I-metric space] An I-metric space (X, d, f) is said to be I-complete if every I-cauchy sequence in X I-converges to some point of X ; otherwise (X, d, f) is called I-incomplete.

Definition (2.8) [I-fixed point] Let X be a non-empty set and $f : X \to X$ is an idempotent map. A map $h : X \to X$ is said to have an I-fixed point x ($\in X$) if $(fh)(x) = f(x)$.

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Definition (2.9) [I-continuity] Let (X, d_1, f), (Y, d_2, g) be two I-generalized metric spaces. Then a function $h : (X, d_1, f) \to (Y, d_2, g)$ is said to be I-continuous at a point $a \in X$, if corresponding to every $\epsilon > 0$, $\exists\ \delta > 0$ such that $d_1(f(x), a) < \delta \implies d_2\big((gh)(x), h(a)\big) < \varepsilon$.
h is said to be I-continuous on *X*, if it is I-continuous at every point of *X*.

Definition (2.10) [I-injective mapping] Let (X, d_1, f), (Y, d_2, g) be two I-generalized metric spaces. A mapping $h : X \to Y$ is said to be I-injective if for all $x_1, x_2 \in X$, $h(x_1) = h(x_2) \Longrightarrow f(x_1) = f(x_2)$.

Definition (2.11) [I-Housdorff space] An I-generalized metric space (*X*, *d*, *f*) is said to be I-Housdorff , if for any two I-distinct points *x*, *y* in *X* , there exists a positive real number *r* such that $S_f(x, r) \cap S_f(y, r) = \Phi$.

Theorem (2.12) Let (*X*, *d*, *f*) be an I-generalized metric space. Then
(i) $d(x, x) = 0$, $\forall x \in X$, $i.e.$, $\forall x, y \in X, x = y \implies d(x, y) = 0$.
(ii) $d\big(x, f(y)\big) = d(f(x), y) = d\big(f(x), f(y)\big) = d\big(f(y), f(x)\big) \geq d(x, y), d(y, x)$, $\forall x, y \in X$.
(iii) $d\big(x, f(x)\big) = 0$, $\forall x \in X$.

Proof: Trivial.

3. Main Results

Theorem (3.1) [Banach contraction principle] Let (X, d, f) be an I-g.m.s. $c \in (0, 1)$ and $h : X \to X$ be a map such that for each $x, y \in X$, $d\big((fh)(x), h(y)\big) \leq c\, d(x, y)$ with $y \neq f(x)$, $x \neq f(y)$, then
(i) there exists a point $a \in X$ such that for each $x \in X$, the sequence $h^n(x)$ I-converges to a .
(ii) $(fh)(a) = f(a)$ and for each $b \in X$, $(fh)(b) = f(b) \Longrightarrow f(a) = f(b)$, $i.e.$, h has an I-unique I-fixed point.

Proof: (i) Let $x \in X$ and consider the sequence $\{h^n(x)\}$. If *x* is a periodic point for *h* , then $h^k(x) = x$ for some $k \in \mathbb{N}$ and then $d\big(x, (fh)(x)\big) = d\Big(h^k(x), (fh^{k+1})(x)\Big) \leq c\, d(h^{k-1}(x), h^k(x))$
$\leq c\, d(h^{k-1}(x), (fh^k)(x)) \leq c^2\, d(h^{k-2}(x), h^{k-1}(x)) \leq c^2 d(h^{k-2}(x), (fh^{k-1})(x))$
$\leq \cdots \ldots \leq c^k d\big(x, h(x)\big) \leq c^k d\big(x, (fh)(x)\big) \Longrightarrow d\big(x, (fh)(x)\big) = 0$ (since $0 < c < 1$).
$\Longrightarrow (fh)(x) = f(x) \Longrightarrow x$ is an I-fixed point of *h* .

Let $h^n(x) \neq h^m(x), \forall m, n \in \mathbb{N}$ with $m \neq n$.
Now $\forall y \in X$, $d\big(y, (fh^4)(y)\big) \leq d\big(f(y), h(y)\big) + d\big((fh)(y), h^2(y)\big) + d(h^2(y), (fh^4)(y))$
$= d\big(y, (fh)(y)\big) + d\big(h(y), (fh^2)(y)\big) + d(h^2(y), (fh^4)(y))$

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$\leq d\big(y,(fh)(y)\big) + c\, d\big(y,(fh)(y)\big) + c^2\, d(y,(fh^2)(y))$
$= (1+c)d\big(y,(fh)(y)\big) + c^2\, d\big(y,(fh^2)(y)\big) = \sum_{i=0}^{2k-3} c^i d\big(y,(fh)(y)\big) + c^{2k-2} d\big(y,(fh^2)(y)\big)$, for $k = 2$.

Let for any ≥ 2 , $d\Big(y,(fh^{2k})(y)\Big) \leq \sum_{i=0}^{2k-3} c^i d\big(y,(fh)(y)\big) + c^{2k-2} d\big(y,(fh^2)(y)\big)$, $\forall y \in X$.

Now $\forall y \in X$, $d\Big(y,(fh^{2k+2})(y)\Big) \leq d\big(f(y),h(y)\big) + d\big((fh)(y),h^2(y)\big) + d(h^2(y),(fh^{2k+2})(y))$
$= d\big(y,(fh)(y)\big) + d\big(h(y),(fh^2)(y)\big) + d(h^2(y),(fh^{2k+2})(y))$
$\leq d\big(y,(fh)(y)\big) + c\, d\big(y,h(y)\big) + c^2 d(y,h^{2k}(y))$

$$\leq d\big(y,(fh)(y)\big) + c\, d\big(y,(fh)(y)\big) + c^2\left(\sum_{i=0}^{2k-3} c^i d\big(y,(fh)(y)\big) + c^{2k-2} d\big(y,(fh^2)(y)\big)\right)$$

$$= \sum_{i=0}^{2k-1} c^i d\big(y,(fh)(y)\big) + c^{2k} d\big(y,(fh^2)(y)\big)$$

Therefore by mathematical induction, we have
$\forall y \in X, d(y,h^{2k}(y)) \leq \sum_{i=0}^{2k-3} c^i d\big(y,(fh)(y)\big) + c^{2k-2} d\big(y,(fh^2)(y)\big), \;\; \forall k \geq 2$ (1)

Similarly, by mathematical induction, we shall get
$d\Big(y,(fh^{2k+1})(y)\Big) \leq \sum_{i=0}^{2k} c^i d\big(y,(fh)(y)\big), \forall y \in X , \forall k \geq 0$ (2)

From (1) and (2) we get
$d\Big(h^n(x),(fh^{n+2k})(x)\Big) \leq c^n d(x,h^{2k}(x)) \leq c^n d(x,(fh^{2k})(x))$

$$\leq c^n \sum_{i=0}^{2k-2} c^i\, max\big\{d\big(x,(fh)(x)\big),\;\; d(x,(fh^2)(x))\big\}$$

$\leq \frac{c^n}{1-c}\; max\big\{d\big(x,(fh)(x)\big),\;\; d(x,(fh^2)(x))\big\}, \forall n \in \mathbb{N}, \forall k \geq 2$ (3)

and $d\Big(h^n(x),(fh^{n+2k+1})(x)\Big) \leq c^n d(x,(fh^{2k+1})(x))$

$$\leq c^n \sum_{i=0}^{2k} c^i\, max\big\{d\big(x,(fh)(x)\big),\;\; d(x,(fh^2)(x))\big\}$$

$\leq \frac{c^n}{1-c}\; max\big\{d\big(x,(fh)(x)\big),\;\; d(x,(fh^2)(x))\big\}, \forall n \in \mathbb{N}, \forall k \geq 0$ (4)

From (3) and (4), we get
$d\big(h^n(x),(fh^{n+m})(x)\big) \leq \frac{c^n}{1-c}\; max\big\{d\big(x,(fh)(x)\big),\;\; d(x,(fh^2)(x))\big\}, \forall n,m \in \mathbb{N}$ (5)

This shows that $\lim_{n\to\infty} d\big(h^n(x),(fh^{n+m})(x)\big) = 0$ so that $\{h^n(x)\}$ is an I-cauchy sequence in X .
But X is I-complete. Therefore there exists a point $a \in X$ such that $\{h^n(x)\}$ I-converges to a .
Now $d\big((fh^{n+1})(x),h(a)\big) \leq cd((fh^n)(a),a) \to 0$ as $n \to \infty$ so that $d\big((fh^{n+1})(x),h(a)\big) \to 0$ as $n \to \infty$.

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Therefore $\{h^n(x)\}$ I-converges to $h(a)$ also.
Therefore $(fh)(a) = f(a)$ so that a is an I-fixed point of h.

Now let $b \in X$ be such that $(fh)(b) = f(b)$. Then
$d(a,b) \leq d\big(f(a), f(b)\big) = d\big((fh)(a),(fh)(b)\big) = \; d\big(h(a),(fh)(b)\big) \leq c\; d(a,b)$
$\Rightarrow d(a,b) = 0$ (since $0 < c < 1$) $\Rightarrow f(a) = f(b)$

Definition (3.2) [T-orbitally I-completeness] Let (X, d, f) be an I-g.m.s. and $T: X \to X$. (X, d, f) is said to be T-orbitally I-complete iff every I-cauchy sequence which is contained in $\{x, T(x), T^2(x), \dots\dots\}$ for some $x \in X$, I-convergence in X .

Definition (3.3) [ε-chainable] An I-g.m.s. (X, d, f) is said to be ε-chainable if for any two points $a, b \in X$, there exists a finite set of points $a = \; x_o, x_1, \dots\dots, x_n = b$ such that $d(f(x_{i-1}),\; x_i) \leq \varepsilon$ for $i = 1,2, \dots\dots, n$, where $\varepsilon > 0$.

Definition (3.4) A mapping $: X \to X$, where (X, d, f) is an I-g.m.s., is called
(i) Locally contractive if for every $x \in X, \exists\; \varepsilon_x > 0$ and $\lambda_x \in [0,1)$ such that
$\forall p, q \in \{y \in X \mid d(x,y) \leq \varepsilon_x\}$, the relation $d(T(p), T(q)) \leq \lambda_x d(p,q)$ holds.
(ii) Locally I-contractive if for every $x \in X, \exists\; \varepsilon_x > 0$ and $\lambda_x \in [0,1)$ such that
$\forall p, q \in \{y \in X \mid d(x,y) \leq \varepsilon_x\}$, the relation $d((fT)(p), T(q)) \leq \lambda_x d(p,q)$ holds.

Definition (3.5) Let (X, d, f) be an I-g.m.s.. Then $T: X \to X$ is called $(\varepsilon\,,\;\lambda)$ uniformly locally I-contractive if it is locally I-contractive at all points $x \in X$ and $\varepsilon,\;\lambda$ do not depend on x , *i.e.* ,
$d(f(x), y) < \varepsilon \Rightarrow d\big((fT)(x), T(y)\big) < \lambda\, d(x,y), \forall x, y \in X$, where $\varepsilon > 0\,,\;\lambda \in [0,1)$.

Note (3.6) From definition (3.5) it is clear that a uniformly locally I-contractive map is I-continuous.

Theorem (3.7) If $T: X \to X$ is an $(\varepsilon\,,\;\lambda)$ uniformly locally I-contractive mapping defined on a T-orbitally I-complete, $\frac{\varepsilon}{2}$ - chainable I-g.m.s. (X, d, f) such that $Tf = fT$ and satisfying the following condition (A) for all $x, y, z \;\in X, d(f(x), y) < \frac{\varepsilon}{2}$ and $d(f(y), z) < \frac{\varepsilon}{2} \Rightarrow d(f(x), z) < \varepsilon$, then T has an I-unique I-fixed point in X.

Proof: Let $\in X$. Since (X, d, f) is $\frac{\varepsilon}{2}$ – chainable and, $T(x) \in X$, there exists finite number of points $x = x_o\,, x_1\,, \dots\dots\,,\; x_n = T(x)$ such that $d(f(x_{i-1}), x_i) < \frac{\varepsilon}{2}$, for $i = 1,2, \dots, n$ (1)
Without loss of generality let $x_1\,, x_2\,, \dots\dots\,, x_{n-1}$ are I-distinct; and if $n > 2$, then assume that $x_o\,, x_1\,, x_2\;, \dots\dots, x_n$ are I-distinct.

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For $n = 1$, from (1) we have $d\big(f(x), T(x)\big) < \frac{\varepsilon}{2}$
For $n = 2$, from (1) and (A) we have $d\big(f(x), T(x)\big) < \frac{2\varepsilon}{2}$
Let $n > 2$ and $n = 2m + 1$ ($m \geq 1$) be odd.
Now $d\big(f(x), T(x)\big) \leq d(f(x), x_1) + d(f(x_1), x_2) + \cdots \ldots + d(x_{2m}, (fT)(x))$
$= d(f(x), x_1) + d(f(x_1), x_2) + \cdots \ldots + d(f(x_{2m}), T(x)) < (2m + 1)\frac{\varepsilon}{2} = \frac{n\varepsilon}{2}$.
Let $n > 2$ and $n = 2m$ ($m \geq 2$) be even. Then
$d\big(f(x), T(x)\big) \leq d(f(x), x_2) + d(f(x_2), x_3) + \cdots \ldots + d(x_{2m-1}, (fT)(x))$
$= d(f(x), x_2) + d(f(x_2), x_3) + \cdots \ldots + d(f(x_{2m-1}), T(x)) < \varepsilon + (2m - 2)\frac{\varepsilon}{2} = \frac{n\varepsilon}{2}$ (by (1) and (A))
Therefore $d\big(f(x), T(x)\big) \leq \frac{n\varepsilon}{2}$ (2)
Since T is (ε , λ) uniformly locally I-contractive, following (1) we have
$d\big((fT)(x_{i-1}), T(x_i)\big) < \lambda\, d(x_{i-1}, x_i) \leq \lambda\, d(f(x_{i-1}), x_i) < \frac{\lambda\varepsilon}{2}$, for $i = 1,2, \ldots \ldots ., n$
Therefore by induction, $\big((fT^m)(x_{i-1}), T^m(x_i)\big) < \frac{\lambda^m \varepsilon}{2}$, $\forall m \in \mathbb{N}$, for $i = 1,2, \ldots \ldots, n$
and $d\big((fT^m)(x_o), T^m(x_2)\big) < \lambda^m \varepsilon$, $\forall m \in \mathbb{N}$ (by (A)).
Therefore following the procedure of proving of (2), we have
$d\big((fT^m)(x), T^{m+1}(x)\big) < \frac{\lambda^m\, n\varepsilon}{2}$, $\forall m \in \mathbb{N}$ (3)
Now let $(fT^m)(x) = (fT^n)(x)$, for some $m, n \in \mathbb{N}$ with $m > n$.
Let $p = m - n$, $u = T^n(x)$.
Then $(fT^p)(u) = f(u)$ so that $(fT^{kp})(u) = f(u)$, $\forall k \in \mathbb{N}$; since $fT = Tf$
Now, $T(u) \in X$. Therefore, similarly we shall get
$d\big((fT^m)(u), T^{m+1}(u)\big) < \frac{\lambda^m\, n\varepsilon}{2}$, $\forall m \in \mathbb{N}$, for some $\in \mathbb{N}$. (4) (similar to (3)).
Then $d\big(f(u), T(u)\big) = d\big(f(u), (fT)(u)\big) = d((fT^{kp})(u), T^{kp+1}(u))$ (since $fT = Tf$)

$$< \frac{\lambda^{kp}\, n\varepsilon}{2} \to 0 \text{ as } k \to \infty \text{ (since } \lambda \in [0,1)).$$

therefore $(fT)(u) = f(u)$.
therefore let $(fT^m)(x) \neq (fT^n)(x)$, $\forall m, n \in \mathbb{N}$.
Now we shall show that $\{T^n(x)\}$ is an I-cauchy sequence in X . Since $\lambda \in [0,1), \exists\, k\ (> 2) \in \mathbb{N}$ such that $\lambda^k < \frac{1}{n}$.
By (3), $d\Big((fT^k)(x), T^{k+1}(x)\Big) < \frac{\lambda^k\, n\varepsilon}{2} < \frac{\varepsilon}{2}$ and $d\Big((fT^{k+1})(x), T^{k+2}(x)\Big) < \frac{\lambda^{k+1}\, n\varepsilon}{2} < \frac{\varepsilon}{2}$
Therefore by (A), $d\Big((fT^k)(x), T^{k+2}(x)\Big) < \varepsilon$ (5)
Let m ($> k$) $\in \mathbb{N}$ be arbitrary.
If $n = 2q + 1$ ($q \geq 0$) be odd, then
$d\big((fT^m)(x), T^{m+n}(x)\big) \leq d\big((fT^m)(x), T^{m+1}(x)\big) + d\big((fT^{m+1})(x), T^{m+2}(x)\big)$

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$$+\cdots\ldots+d\big((fT^{m+2q})(x),T^{m+2q+1}(x)\big)$$

$< (\lambda^m+\lambda^{m+1}+\cdots\ldots+\lambda^{m+2q})\frac{n\varepsilon}{2}$ (by (3)) $< \frac{\lambda^m}{1-\lambda}\ \frac{n\varepsilon}{2}$

If $n=2q$ $(q\ \geq 1)$ be even, then

$$d\big((fT^m)(x),T^{m+n}(x)\big)\leq d\big((fT^m)(x),T^{m+2}(x)\big)+d\big((fT^{m+2})(x),T^{m+3}(x)\big)$$
$$+\cdots\ldots+d\big((fT^{m+2q-1})(x),T^{m+2q}(x)\big)$$

$< \lambda^{m-k}\varepsilon+(\lambda^{m+2}+\lambda^{m+3}+\cdots\ldots+\lambda^{m+2q-1})\frac{n\varepsilon}{2}$ (by (A)) $< \lambda^{m-k}\varepsilon+\frac{\lambda^{m+2}}{1-\lambda}\ \frac{n\varepsilon}{2}$ (by (3) and (4))

$$=\frac{\lambda^{m-k}\varepsilon}{2(1-\lambda)}\ (2-2\lambda+n\lambda^{k-2})$$

Therefore for positive integer *n*, we have

$d\big((fT^m)(x),T^{m+n}(x)\big)< \frac{\lambda^{m-k}\varepsilon}{2(1-\lambda)}\ r$, where $r=max\{n\ \lambda^k\ ,2-2\ \lambda+n\ \lambda^{k-2}\}$

Since *k* is fixed and $\in[0,1)$, hence $\lambda^{m-k}\to 0$, as $m\ \to\ \infty$, so that

$d\big((fT^m)(x),T^{m+n}(x)\big)\to 0$ as $\to\infty$. Therefore $\{T^n(x)\}$ is I-cauchy.

Since *X* is *T*-orbitally I-complete, hence $\{T^n(x)\}$ I-converges to some point *u* in *X*.

Since *T* is uniformly locally I-contractive map, hence *T* is I-continuous.

Therefore $\{T(T^n(x))\}$ I-converges t u. .

Again $\{T(T^n(x))\}$, $i.e.,\{T^{n+1}(x)\}$ I-converges to *u*.

Therefore $(fT)(u)=f(u)$ (6)

Therefore *u* is an I-fixed point of *T* .

Let *v* be another I-fixed point of *T*. then $(fT)(v)=f(v)$ (7)

Since *X* is $\frac{\varepsilon}{2}$–chainable, there exists finite number of points $u=x_o\ ,x_1\ ,\ldots\ldots\ \ ,x_n=v$ such that $d(f(x_{i-1}),x_i)< \frac{\varepsilon}{2}$, for $i=1,2,\ldots\ldots.,n$.

Then similarly, as proved above, we have $\big((fT^m)(u),T^m(v)\big)< \frac{\lambda^m n\varepsilon}{2}$,$\forall m\in\mathbb{N}$ (8)

Therefore $d(f(u),v)=d\big(f(u),f(v)\big)=\ d\big((fT^m)(u),(fT^m)(v)\big)$

((6), (7), and since $T=Tf$)

$$=d\big((fT^m)(u),T^m(v)\big)< \frac{\lambda^m n\varepsilon}{2}\ \text{(by (8))}\to 0 \text{ as } m\to\infty$$

Therefore $d(f(u),v)=\ 0$ so that $f(u)=f(v)$. Therefore *u* is I-unique.

Definition (3.8) [Sequentially convergence] Let (X,d,f) be an I-g.m.s. A map $T:X\ \to X$ is said to be sequentially I-convergent if for every sequence $\{y_n\}$, if $\{T(x_n)\}$ is I-convergent, then $\{y_n\}$ is also I-convergent.

T is said to be sub-sequentially I-convergent if for every sequence $\{y_n\}$, if $\{T(x_n)\}$ is I-convergent, then $\{y_n\}$ has an I-convergent subsequence.

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Theorem (3.9)[Extended Kannan contraction principle] Let (X, d, f) be an I-complete I-g.m.s. and $T, S : X \to X$ such that T is I-continuous, sub-sequentially I-convergent and fT is I-injective .
If $\lambda \in [0, \ 1/2)$ and $d\big((fTS)(x), (TS)(y)\big) \le \lambda\big(d\big((fT)(x), (TS)(x)\big) + d\big((fT)(y), (TS)(y)\big)\big), \forall x, y \in X$ (1)
Then *S* has an I-fixed point. In addition, if $fS = Sf$ then this I-fixed point of *S* is I-unique. Also if *T* is sequentially I-convergent then for every $x_o \in X$ the sequence $\{S^n(x_o)\}$ I-converges to this I-fixed point.

Proof: Let $x_o \in X$ be arbitrary. For all $n \in \mathbb{N}$ let $x_n = S^n(x_o)$. Using (1) we get

$$d\big((fT)(x_n), T(x_{n+1})\big) = d((fTS)(x_{n-1}), (TS)(x_n))$$
$$\le \lambda\big(d\big((fT)(x_{n-1}), (TS)(x_{n-1})\big) + d\big((fT)(x_n), (TS)(x_n)\big)\big) \quad (2)$$
$$\Rightarrow d\big((fT)(x_n), T(x_{n+1})\big) \le \frac{\lambda}{1-\lambda} d\big((fT)(x_{n-1}), T(x_n)\big) \quad (3)$$
$$\le \left(\frac{\lambda}{1-\lambda}\right)^2 d\big((fT)(x_{n-2}), T(x_{n-1})\big) \text{ (by the same argument)}$$
$$\le \cdots \dots \dots \dots \le \left(\frac{\lambda}{1-\lambda}\right)^n d\big((fT)(x_o), T(x_1)\big) \quad (4)$$

Now for all $m, n \in \mathbb{N}$, we have

$$d\big((fT)(x_m), T(x_n)\big) = d((fTS)(x_{m-1}), (TS)(x_{n-1}))$$
$$\le \lambda\left(d\big((fT)(x_{m-1}), (TS)(x_{m-1})\big) + d\big((fT)(x_{n-1}), (TS)(x_{n-1})\big)\right) \text{ (by (1))}$$
$$= \lambda\left(d\big((fT)(x_{m-1}), T(x_m)\big) + d\big((fT)(x_{n-1}), T(x_n)\big)\right)$$
$$\le \lambda\left(\left(\frac{\lambda}{1-\lambda}\right)^{m-1} + \left(\frac{\lambda}{1-\lambda}\right)^{n-1}\right) d\big((fT)(x_o), T(x_1)\big) \text{ (by (4))} \to 0 \text{ as } m, n \to \infty \text{ (since } 0 \le \lambda < 1/2 \text{)}$$

Therefore $\{T(x_n)\}$ is an I-cauchy sequence. Since *X* is I-complete, hence $\{T(x_n)\}$ I-converges to some point $v \in X$. Since *T* is sub-sequentially I-convergent, the sequence $\{x_n\}$ has an I-convergent sub-sequence $\{x_{n_k}\}$ I-converging to a point $\in X$. Since *T* is I-continuous and $\{x_{n_k}\}$ I-converges to *u* , hence $\{T(x_{n_k})\}$ I-converges to $T(u)$.
Again $\{T(x_n)\}$ I-converges to *v* $\Rightarrow \{T(x_{n_k})\}$ I-converges to *v*. Therefore $T(u)$ and *v* are I-unique, so that $(fT)(u) = f(v)$.

Now
$$d\big((fTS)(u), T(u)\big) \le d\big((fTS)(u), (TS^{n_k})(x_o)\big) + d((fTS^{n_k})(x_o), \ (TS^{n_k+1})(x_o)) + d((TS^{n_k+1})(x_o), \ (fT)(u)) .$$
$$\le \lambda\left(d\big((fT)(u), (TS)(u)\big) + d\left((fT)\big(S^{n_k-1}(x_o)\big), (TS^{n_k})(x_o)\right)\right) + \left(\frac{\lambda}{1-\lambda}\right)^m d\big((fT)(x_o), T(x_1)\big) + d((fT)\big(x_{n_k+1}\big), T(u)) \text{ (by (1) and (4)).}$$
$$= \lambda d\big((fTS)(u), T(u)\big) + \lambda d\left((fT)\big(x_{n_k-1}\big), T\big(x_{n_k}\big)\right) + \left(\frac{\lambda}{1-\lambda}\right)^m d\big((fT)(x_o), T(x_1)\big)$$

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$$+d((fT)(x_{n_k+1}),T(u)) \quad (5)$$

Therefore $d\big((fTS)(u),T(u)\big) \le \frac{\lambda}{1-\lambda} d\left((fT)(x_{n_k-1}),T(x_{n_k})\right) + \frac{\lambda}{1-\lambda}\left(\frac{\lambda}{1-\lambda}\right)^m d\big((fT)(x_o),T(x_1)\big)$

$$+\frac{\lambda}{1-\lambda} d((fT)(x_{n_k+1}),T(u)) \to 0 \text{ as } k \to \infty$$

(since $\{T(x_n)\}$ is I-cauchy, $0 \le \lambda < 1/2$ and $\{T(x_{n_k})\}$ I-converges to $T(u)$) (6)

therefore $d\big((fTS)(u),T(u)\big) = 0$ (7)

therefore $(fTS)(u) = (fT)(u)$ (8)

$\Longrightarrow (fS)(u) = f(u)$ (since fT is I-injective) (9)

Therefore u is an I-fixed point of S.

Let $Sf = fS$. Let w be any I-fixed point of S in X. then $(fS)(w) = f(w)$ (10)

from (10) we get $(fTfS)(w) = (fTf)(w) = (fTSf)(w)$ (since $Sf = fS$) (11)

Now $d\big((fTS)(u),(TS)(f(w))\big) \le \lambda\left(d\big((fT)(u),(TS)(u)\big) + d\big((fT)(f(w)),(TS)(f(w))\big)\right)$ (by (1))

$= \lambda\big(d\big((fTS)(u),T(u)\big) + d\big((fT)(f(w)),(TS)(f(w))\big) = 0\big)$ (by (8) and (11))

Therefore $d\big((fTS)(u),(TSf)(w)\big) = 0$

$\Longrightarrow (fTS)(u) = (fTSf)(w) \Longrightarrow (fS)(u) = (fSf)(w)$ (Since fT is I-injective)

$\Longrightarrow (fS)(u) = (fS)(w)$ (since $f = fS$)

$\Longrightarrow f(u) = f(w)$ (by (9) and (10)).

therefore u is I-unique.

Now if T is sequentially I-convergent, replacing $\{n_k\}$ by $\{n\}$, we can say that $\{x_n\}$ I-converges to u, *i.e.* , $\{x_n\}$ I-converges to the I-fixed point of S in X .

Definition (3.10) Let (X,d,f) be an I-g.m.s. and $: X \to X$. We say that x $(\in X)$ is an I-periodic point of T if $(fT^k)(x) = f(x)$ for some $k \in \mathbb{N}$.

Theorem (3.11) let (X,d,f) be an I-Housdorff and I-complete I-g.m.s. Let $T : X \to X$ such that $fT = Tf$ and for all $, y \in X$,

$$d\big((fT)(x),T(y)\big) \le \frac{1}{2}\Big(d\big(f(x),T(x)\big) + d\big(f(y),T(y)\big)\Big) - \varphi(d(f(x),T(x)),d(f(y),T(y))) \quad (1)$$

where $\varphi : [0,\infty) \times [0,\infty) \to [0,\infty)$ is continuous and $\varphi(a,b) = 0$ iff $a = b = 0$. Then there exists an I-unique I-fixed point of T in X .

Proof: Let $x_o \in X$ be arbitrary. let $x_n = T(x_{n-1}) = T^n(x_o), \forall n \in \mathbb{N}$.

If for some $n \in \mathbb{N}$, $f(x_n) = f(x_{n-1})$, the proof is finished.

Let $f(x_n) = f(x_{n-1}), \ \forall n \in \mathbb{N}$.

From (1) we get

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$d(f(x_{n+1}),x_n) = d((fT)(x_n),T(x_{n-1}))$
$\leq \frac{1}{2}\Big(d\big(f(x_n),T(x_n)\big) + d\big(f(x_{n-1}),T(x_{n-1})\big)\Big) - \varphi\left(d\big(f(x_n),T(x_n)\big), d\big(f(x_{n-1}),T(x_{n-1})\big)\right)$ (2)

$\leq \frac{1}{2}\Big(d\big(x_n, f(x_{n+1})\big) + d\big(x_{n-1}, f(x_n)\big)\Big)$
$\Longrightarrow d(f(x_{n+1}),x_n) \leq d(f(x_n),x_{n-1}), \forall n \in \mathbb{N}$
Therefore the sequence $\{d(f(x_{n+1}),x_n)\}$ is monotone decreasing and bounded below, and hence is convergent in $\mathbb{R}$. Therefore there exists $p \geq 0$ such that $\lim_{n\to\infty} d(f(x_{n+1}),x_n) = p$.
Taking $n \to \infty$, from (2) we get
$p \leq \frac{1}{2}(p+p) - \varphi(p,p)$ (since φ is continuous) .
This implies that $0 \leq \varphi(p,p) \leq 0 \Longrightarrow \varphi(p,p) = 0$ so that $p = 0$.
Therefore $\lim_{n\to\infty} d(f(x_{n+1}),x_n) = 0$ (3)
Using (1), (3) and continuity and given property of φ , similarly, we shall get
$\lim_{n\to\infty} d(f(x_{n+2}),x_n) = 0$ (4)
We shall now show that *T* has an I-periodic point. If possible, let *T* has no I-periodic point.
Then $f(x_n) \neq f(x_m)$, $\forall m,n \in \mathbb{N}$ with $m \neq n$.
In this case, we claim that $\{x_n\}$ is I-cauchy. if possible let $\{x_n\}$ is not I-cauchy. Then there exists $\varepsilon > 0$ such that for a positive integer *k*, there exists integer n_k $(> k)$ and the least positive integer m_k with $m_k > n_k > k$ such that $d\big(f(x_{n_k}),x_{m_k}\big) > \varepsilon$ (5)
Then $d\big(f(x_{n_k}),x_{m_k-1}\big) \leq \varepsilon$ (6)
Now $\varepsilon < d\big(f(x_{n_k}),x_{m_k}\big) \leq d\big(f(x_{m_k}),x_{m_k-2}\big) + d\big(f(x_{m_k-2}),x_{m_k-1}\big) + d\big(f(x_{m_k-1}),x_{n_k}\big)$
$\leq d\big(f(x_{m_k}),x_{m_k-2}\big) + d\big(f(x_{m_k-2}),x_{m_k-1}\big) + \varepsilon$ (by (5) and (6))
Then using (3) and (4), we get $\lim_{k\to\infty} d\big(f(x_{n_k}),x_{m_k}\big) = \varepsilon$ (7)
now $d\big(f(x_{m_k}),x_{n_k}\big) = d\big((fT)(x_{m_k-1}),T(x_{n_k-1})\big)$
$\leq \frac{1}{2}\Big(d\big(f(x_{m_k-1}),x_{m_k}\big) + d\big(f(x_{n_k-1}),x_{n_k}\big)\Big) - \varphi\left(d\big(f(x_{m_k-1}),x_{m_k}\big), d\big(f(x_{n_k-1}),x_{n_k}\big)\right)$
taking $k \to \infty$ we get
$\varepsilon \leq \frac{1}{2}(0+0) - \varphi(0,0)$ (by (3) and (7) and continuity of φ)
Therefore $\varepsilon \leq 0$ (by property of φ), a contradiction.
Therefore $\{x_n\}$ is I-cauchy in *X*. Since *X* is I-complete, $\{x_n\}$ I-converges to some point *u* in *X* .
Now $d\big((fT)(x_n),T(u)\big) \leq \frac{1}{2}\Big(d\big(f(x_n),T(x_n)\big) + d\big(f(u),T(u)\big)\Big) -$
$\varphi\Big(d\big(f(x_n),T(x_n)\big), d(f(u),T(u))\Big)$
(by (1))

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$$\Rightarrow d\big(f(x_{n+1}),T(u)\big) \le \frac{1}{2}\Big(d(f(x_n),x_{n+1}) + d\big(f(u),T(u)\big)\Big)$$
$$-\varphi\left(d(f(x_n),x_{n+1}), d\big(f(u),T(u)\big)\right) \qquad (8)$$
$$\Rightarrow d\big(f(x_{n+1}),T(u)\big) \le \frac{1}{2}\Big(d(f(x_n),x_{n+1}) + d\big(f(u),T(u)\big)\Big)$$
$\Rightarrow \lim_{n\to\infty} d\big(f(x_{n+1}),T(u)\big) \le \frac{1}{2} d(f(u),T(u))$ (by (3)) (9)

Let $f(x_n) \neq f(u)$ and $(x_n) \neq (fT)(u), \forall n \ge 2$.

Then $d\big(f(u),T(u)\big) \le d(f(u),x_n) + d(f(x_n),x_{n+1}) + d(f(x_{n+1}),T(u))$

$\Rightarrow d(f(u),T(u)) \le \lim_{n\to\infty} d\big(f(x_{n+1}),T(u)\big)$ (10)

(by (3) and since $\{x_n\}$ I-converges to u)

From (9) and (10) we get

$$d(f(u),T(u)) \le \lim_{n\to\infty} d\big(f(x_{n+1}),T(u)\big) \le \frac{1}{2} d(f(u),T(u))$$
$$\Rightarrow d\big(f(u),T(u)\big) = 0 \Rightarrow (fT)(u) = f(u)$$

Therefore u is an I-fixed point of T so that u is an I-periodic point of T which contradicts the fact that T has no I-periodic point.

Let for some positive integer $r \ge 2$, $f(x_r) = f(u)$ or $f(x_r) = (fT)(u)$

Since T has no I-periodic point, hence $f(u) \neq f(x_o)$

Now $d\big((fT^n)(u),u\big) = d\big((fT^n)(x_r),u\big) = d(f(x_{n+r}),u)$

Or, $d\big((fT^n)(u),u\big) = d\big((fT^{n-1})(x_r),u\big) = d(f(x_{n+r-1}),u), \forall n \in \mathbb{N}$ (since $fT = Tf$).

Since r (≥ 2) is fixed, hence $\{x_{n+r}\}$ and $\{x_{n+r-1}\}$ are subsequences of $\{x_n\}$ and since $\{x_n\}$ I-converges to u in X which is I-Housdorff, hence $\{x_{n+r}\}$ and $\{x_{n+r-1}\}$ both I-converges to u .

Therefore $\lim_{n\to\infty} d(f(x_{n+r}),u) = 0 = \lim_{n\to\infty} d(f(x_{n+r-1}),u)$.

Therefore $\lim_{n\to\infty} d\big((fT^n)(u),u\big) = 0$ (since $fT = Tf$) (11)

Again since X is I-Housdorff and T has no I-periodic point, from (11) we have

$\lim_{n\to\infty} d\big((fT^{n+2})(u),u\big) = 0$, (since $fT = Tf$). (12)

Since T has no I-periodic point, hence $(fT^s)(u) \neq (fT^t)(u), \forall s,t \in \mathbb{N}$ with $s \neq t$ (13)

Using (13) and rectangular inequality, we have

$$\left|d\big((fT^{n+1})(u),T(u)\big) - d(f(u),T(u))\right| \le d\big((fT^{n+1})(u),T^{n+2}(u)\big) + d\big((fT^{n+2})(u),u\big)$$

taking $n \to \infty$, we have $\lim_{n\to\infty} d\big((fT^{n+1})(u),T(u)\big) = d(f(u),T(u))$ (14)

(Using (3) and replacing x_o by u)

Similarly we have $\lim_{n\to\infty} d\big((fT^n)(u),T(u)\big) = d(f(u),T(u))$ (15)

Now $d\big((fT^{n+1})(u),T(u)\big)$

$\le \frac{1}{2}\Big(d\big((fT^n)(u),T(u)\big) + d(f(u),T(u))\Big) - \varphi\Big(d\big((fT^n)(u),T(u)\big), d(f(u),T(u))\Big)$ (by (1))

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Taking $n \to \infty$, we get

$$d(f(u),T(u)) \le \frac{1}{2}\Big(d(f(u),T(u)) + d(f(u),T(u))\Big) - \varphi\Big(d(f(u),T(u)), d(f(u),T(u))\Big)$$

(by (14) , (15) and continuity of φ)

$\Rightarrow 0 \le \varphi\Big(d(f(u),T(u)), d(f(u),T(u))\Big) \le 0 \Rightarrow \varphi\Big(d(f(u),T(u)), d(f(u),T(u))\Big) = 0$

$\Rightarrow d(f(u),T(u)) = 0$ (by property of φ)

$\Rightarrow (fT)(u) = f(u) \Rightarrow u$ is an I-fixed point of T .so that u is an I-periodic point of T , a contradiction.

Therefore T has an I-periodic point. Therefore there exists $u \in X$ such that

$(fT^k)(u) = f(u)$, for some $k \in \mathbb{N}$ (16)

If $k = 1$ in (16), then $(fT)(u) = f(u)$ and in this case u is an I-fixed point of T.

Let $k > 1$. Let $(fT^k)(u) \neq (fT^{k-1})(u)$. Then $d\Big((fT^{k-1})(u), T^k(u)\Big) > 0$ so that

$\varphi\Big(d\Big((fT^{k-1})(u), T^k(u)\Big), d\Big((fT^{k-1})(u), T^k(u)\Big)\Big) > 0$

Now $d(f(u),T(u)) = d\Big((fT^k)(u), (fT)(u)\Big) = d\Big((fT^k)(u), (fT^{k+1})(u)\Big)$ (by (16) and $T = Tf$)

$$= d\Big((fT)(T^{k-1}(u)), T(T^k(u))\Big) \le \frac{1}{2}(d((fT^{k-1})(u), T^k(u))) + d((fT^k)(u), T(T^k(u))))$$
$$-\varphi(d((fT^{k-1})(u), T^k(u))), d((fT^k)(u), T(T^k(u))))$$

$< \frac{1}{2}(d((fT^{k-1})(u), T^k(u))) + d(f(u),T(u)))$ (by (16) and $fT = Tf$)

Therefore $d(f(u),T(u)) < d((fT^{k-1})(u), T^k(u)))$ (17)

Again $d((fT^{k-1})(u), T^k(u))) = d\Big((fT)(T^{k-2}(u)), T(T^{k-1}(u))\Big)$

$$\le \frac{1}{2}\Big(d\Big((fT^{k-2})(u), T^{k-1}(u)\Big) + d\Big((fT^{k-1})(u), T^k(u)\Big)\Big)$$
$$-\varphi\Big(d\Big((fT^{k-2})(u), T^{k-1}(u)\Big), d\Big((fT^{k-1})(u), T^k(u)\Big)\Big) \text{ (by (1))}$$

$$\le \frac{1}{2}\Big(d\Big((fT^{k-2})(u), T^{k-1}(u)\Big) + d\Big((fT^{k-1})(u), T^k(u)\Big)\Big)$$

$\Rightarrow d\Big((fT^{k-1})(u), T^k(u)\Big) \le d\Big((fT^{k-2})(u), T^{k-1}(u)\Big)$ (18)

From (17) and (18) we get $(f(u),T(u)) \le d\Big((fT^{k-2})(u), T^{k-1}(u)\Big)$ (19)

Proceeding in this way, after finite number of steps, we get

$d(f(u),T(u)) < d(f(u),T(u)))$, a contradiction.

Therefore $(fT^k)(u) = (fT^{k-1})(u)$

$\Rightarrow (fT)\Big(T^{k-1}(u)\Big) = f\Big(T^{k-1}(u)\Big) \Rightarrow T^{k-1}(u)$ is an I-fixed point of T .

Let there are two points $x, y \in X$ such that $(fT)(x) = f(x)$ and $(fT)(y) = f(y)$.

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Now $d(f(x),y) = d\big(f(x),f(y)\big) = d\big((fT)(x),(fT)(y)\big) = d((fT)(x),T(y))$
$\leq \frac{1}{2}\Big(d\big((fT)(x),T(x)\big) + d((fT)(y),T(y))\Big) - \varphi\Big(d\big((fT)(x),T(x)\big),d((fT)(y),T(y))\Big)$ (by (1))
$= 0$
$\Longrightarrow d(f(x),y) = 0 \Longrightarrow f(x) = f(y)$.
Therefore T has an I-unique I-fixed point in X .

Conclusion

Further study may be continued for generalization of various contractive conditions and fixed point result.

References

[1] Arshad M., Ahmad J.; Karapinar E., Some Common Fixed Point Results in Rectangular Metric Spaces, Intern. J. Analysis, Article ID 307234. (2013).
[2] Aydi H., Karapinar E.; Lakzian H., Fixed Point Results on a class of generalized metric spaces, Math. Sci. (A Springer Open J.), 2012, 6:46.
[3] Banach S. (1922); Sur les operations dans les ensembles abstracts ET leur applications aux equations integrals, fund.math3, pp. 133-181.
[4] Branciari A.; A fixed point theorem of Banach-Caccioppoli type on a class of generalized metric spaces, Publ. Math. Debrecen, (57) 1-2 (2000), PP 31-37.
[5] Das P. and Dey L. K.; A Fixed Point Theorem in a Generalized Metric Space, Soochow J. Math., Vol. 33, No. 1 (2007), PP 33-39.
[6] Kannan R.; Some results on fixed points, Bull. Calcutta Math Soc. 60 (1968), 71-76.
[7] Kikina L., Kikina K.; A fixed point theorem in generalized metric spaces, Demonstratio Mathematica, Vol. XLVI, No. 1, 2013 (10.1515/dema-2013-0432).
[8] Moradi S.; Kannan fixed-point theorem on complete metric spaces and on generalized metric spaces depended on another function, 2009, arXiv: 0903.1577v1.